JOURNAL OF APPROXIMATION THEORY 19, 239–252 (1977)

On the Lax Equivalence Theorem Equipped With Orders

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Communicated by Oved Shisha

Received November 3, 1975

1. INTRODUCTION

The Lax equivalence theorem on the convergence of the solution of the discrete problem to the solution of the given properly posed continuous initial-value (Cauchy) problem states that stability of the finite difference scheme is necessary and sufficient for convergence, provided it is consistent. This theorem, which plays a basic role in the subject, was first established in an operator-theoretic setting in Banach spaces by Peter Lax (in a seminar talk at New York University in January 1954; published with Richtmyer[1] [17] in 1956; see the presentation in Richtmyer and Morton [21, p. 39–59]). It has been generalized in various directions in the meantime. For example, Schultz [23] established an analog for locally convex topological spaces, Ansorge [1], Ansorge and Haas [2], and Ansorge and Geiger [3] obtained certain nonlinear analogs (the latter paper is in the framework of approximation theory), Wendroff [34] gave a strengthened form with a shorter proof based upon Fourier transforms and multipliers, Hersh [13] considered an extension for mixed initial-boundary value problems, and Thompson [32], and Stetter [26] considered the matter for inhomogeneous problems and some functional differential equations. Chartres and Stepleman [10] gave an abstract version with a simple proof, applicable, as they state, to any numerical computation method (they apply it to ordinary differential equations); Stummel, in a series of papers (see, e.g., [27]) presented a more abstract version as part of his general theory of discrete convergence of operators. For results of the Russian school see, for example, Samarskii [22].

The Lax's theorem, just as the Banach–Steinhaus theorem on sequences of linear operators with which it is connected, is a pure convergence theorem. During a colloquium lecture held at the Oxford University Computing

[1] Chartres and Stepleman [10] state that the Lax theorem, also known as the Lax–Richtmyer theorem, is attributed to Kantorovitch but do not refer to any literature. They have however in mind the latter's survey paper [14]. Marinescu in his very recent book [19] cites Kantorovitch (but not Lax). It is interesting to note that Marinescu treats the matter in the framework of Stummel's theory.

ISSN 0021-9045

Laboratory by one of the authors on May 10, 1973, on an application of the Banach–Steinhaus theorem equipped with a rate of convergence to quadrature formulas (see Butzer *et al.* [9]), B. Noble raised the question whether similar methods could be used to furnish Lax's theorem with orders. in the sense that consistency, stability, and convergence would be considered with orders. Already Peetre and Thomée [20] have shown by methods of interpolation space theory that for consistency of order $\alpha > 0$ stability (of order zero) implies convergence with order α in Sobolev spaces; this work was carried on in many directions by Thomée and his collaborators and others independently, for example, by Brenner, Kreis, Löfström, Wahlbin, and Widlund in [4–7, 15, 16, 18, 28–31, 35].

But the original question of Noble whether the Lax theorem in its sufficient as well as necessary part formulation in the setting of general Banach spaces attached to the Cauchy problem, could be generalized to one containing orders, seems apparently not to have been considered. Since that time the matter has been considered at Aachen. Esser [12] studied the problem in the frame of Stummel's theory [27], applying it to a variety of problems, including rate of convergence for Romberg and general Hermite procedures (the error estimates being free of derivatives), for boundary value problems for ordinary differential equations, for Runge–Kutta procedures for initial-value problems for ordinary differential equations, and for integral equations.

It will be the purpose of this note to answer the question for a properly posed initial-value problem, at the same time weakening stability to stability with order (= instability with polynomial order). This material is treated in Section 2. In Section 3 our general theorem is applied to a particular partial differential equation with corresponding difference schemes. Wendroff [34] and Thomée [28] have pointed out the close connection between various definitions of properly posedness and corresponding ones of stability. Therefore it is important to know that there exist examples of initial-value problems satisfying the conditions of our main theorem, Theorem 1, in particular those that are properly posed and at the same time stable of order β with $\beta > 0$. This is the case for $L^p(\mathbb{R})$ with $p \neq 2$ of an example discussed by Brenner-Thomée [4], also treated in Section 3.

Actual interpolation space methods will not be used in our proofs; instead, the estimates are expressed more directly in terms of the K-functional. Then use is made of the fact that this K-functional can be estimated by means of the generalized modulus of continuity in many particular cases (such as $X = L^p(\mathbb{R})$, $1 \leqslant p < \infty$). For this reason our approach is very elementary; for completeness sake the connection with interpolation space theory is mentioned (for the latter see, for example, Butzer and Berens [8, Chap. 3]), and the results of Peetre and Thomée [20], for example, are seen to follow from ours. See also Butzer *et al.* [9a].

2. Generalization of the Lax Theorem

2.1. *Definitions and Basic Results*

Consider the initial-value problem

$$d/dt\, u(t) = Au(t), \qquad t \geqslant 0 \tag{2.1}$$

$$u(0) = f, \qquad f \in X, \tag{2.2}$$

where A is a closed linear operator (independent of t) with domain $D(A)$ dense in the Banach space X (with norm $\|\cdot\|_X$), and f is a given element in X describing the initial state. The problem is to find a one-parameter family $u(t) \in X$, $t \geqslant 0$, satisfying (2.1) and (2.2). It is said to be *properly* (*or correctly*) *posed* if:

(2.3) it has a unique solution for each $f \in U$, U being dense in X, i.e., there exists a one-parameter family of operators $E_0(t)$; $U \to X$ for $t \geqslant 0$ such that $u(t) = E_0(t) f$ for $f \in U$, $t \geqslant 0$, is the solution of (2.1), (2.2), and $E_0(t) f \in D(A)$ for $t \geqslant 0$ (implying $U \subseteq D(A)$),

(2.4) $E_0(t)$ is uniformly continuous for $0 \leqslant t \leqslant T$ ($T > 0$ arbitrarily fixed), i.e.,

$$\|E_0(t) f - E_0(t) g\|_X \leqslant C_T \|f - g\|_X \qquad (f, g \in U;\ 0 \leqslant t \leqslant T),$$

C_T being some positive constant depending on T.

Let $E(t)$ denote the unique continuous extension of $E_0(t)$ from U to the whole space X. It can readily be shown that the solution operators $E(t)$ of a properly posed initial-value problem (2.1), (2.2) form a semigroup of class (C_0) (i.e., for which $E(0) = I\,(=\text{identity})$, $E(t+s) = E(t)\,E(s)$ for all t, $s \geqslant 0$, $\lim_{t\to 0} \|E(t) f - f\|_X = 0$ for $f \in X$) with infinitesimal generator A, and conversely (see Butzer and Berens [8] for basic properties on semigroups).

Let E_k be a finite difference operator so that an iterated application of E_k to f yields $E_k^n f$ which approximates $E(k)^n f = E(nk) f$. With this purpose in mind the following definitions are standard; see [21, 20, 28].

The family of operators $\{E_k\}_{k>0} \subset [X]$ is said to be *consistent of order* $\alpha > 0$ (or possess order of accuracy α) on U with the family $\{E(t)\}_{t>0}$ provided for $T > 0$ there exist positive constants C_T[2] and δ ($\leqslant T$) such that

$$\|[E_k - E(k)]\, E(t) f\|_X \leqslant C_T k^{1+\alpha} B(f) \qquad (0 < k \leqslant \delta,\ 0 \leqslant t \leqslant T;\ f \in U), \tag{2.5}$$

[2] The constant C_T (depending on T) may have different values at each occurrence.

$B(f)$ being some functional (a seminorm) defined on U. Ordinary consistency corresponds to the case that the right side of (2.5) is replaced by $o(k)$ for $k \to 0$.

The family $\{E_k\}_{k>0}$ is said to be *convergent with order* $\alpha > 0$ on a subspace Z of X, if

$$\| E_k^n f - E(nk) f \|_X \leqslant C_T k^\alpha B'(f) \qquad (0 < k \leqslant \delta, 0 \leqslant nk \leqslant T; f \in Z), \quad (2.6)$$

$B'(f)$ being some functional defined on Z. Pure convergence is of course to be understood in the sense that for any $f \in X$

$$\lim_{k \to 0} \| E_k^n f - E(nk) f \|_X = 0 \qquad (0 < k \leqslant \delta, 0 \leqslant nk \leqslant T). \quad (2.7)$$

The family $\{E_k\}_{k>0}$ is said to be *stable of order* $\beta \geqslant 0$, if

$$\| E_k^n f \|_X \leqslant C_T k^{-\beta} \| f \|_X \qquad (0 < k \leqslant \delta, 0 \leqslant nk \leqslant T; f \in X). \quad (2.8)$$

(By the uniform boundedness theorem (2.8) is equivalent to $\| E_k^n f \|_X \leqslant C_T k^{-\beta} M(f)$ for some constant $M(f)$ depending on f. The case $\beta = 0$ is ordinary stability.) As [21, p. 95] notes, this definition agrees with the empirical observation that instability is usually distinguished from ordinary stability by an exponential rather than polynomial growth of error.

In this notation the Lax equivalence theorem in its standard form reads:

Given a properly posed initial-value problem (2.1), (2.2) *in* X *and a finite difference approximation generated by* E_k *satisfying the ordinary consistency condition, then stability* (*of order* O) *is a necessary and sufficient condition for* (*pure*) *convergence.*

In order to equip this theorem with orders, first note that if the family $\{E_k\}$ is stable of order $\beta \geqslant 0$ and consistent of order α on U with $\alpha > \beta$, then $E_k^n f$ converges to $E(nk) f$ with order $\alpha - \beta$ for all $f \in U$. Indeed, since (2.1), (2.2) is properly posed,

$$\begin{aligned} \| E_k^n f - E(nk) f \|_X &= \Big\| \sum_{j=0}^{n-1} E_k^j [E_k - E(k)] \, E((n-1-j) k) f \Big\|_X \\ &\leqslant \sum_{j=0}^{n-1} \{ \| E_k^j \|_{[X]} k^\beta \} k^{-\beta} \| [E_k - E(k)] \, E((n-1-j) k) f \|_X \\ &\leqslant \sum_{j=0}^{n-1} C_T k^{-\beta} k^{1+\alpha} C_T B(f) \\ &\leqslant nk C_T k^{\alpha - \beta} B(f) \qquad (0 < k \leqslant \delta, 0 \leqslant nk \leqslant T; f \in U). \end{aligned} \quad (2.9)$$

In the case of ordinary stability, order of convergence and consistency are both equal to another, namely to α. Further note that the estimate (2.9) is also valid for $0 < \alpha \leqslant \beta$. Although one does not have convergence in this instance, (2.9) may then be interpreted as a restriction upon the growth of error.

The next step is to estimate the rate of convergence for generalized solutions of (2.1), (2.2), i.e., solutions with initial values $f \in X$.

PROPOSITION 1. *Given a properly posed initial-value problem* (2.1), (2.2) *and a family of finite difference operatots* $\{E_k\} \subset [X]$ *consistent of order* $\alpha > 0$ *on* U *with* $\{E(t)\}$.

If the family $\{E_k\}$ *is stable of order* $\beta \geqslant 0$, *then there is a constant* C_T *such that for any* $f \in X$

$$\| E_k^n f - E(nk) f \|_X \leqslant C_T k^{-\beta} K_B(nk^{\alpha-1}, f; X, U) \qquad (0 < k \leqslant \delta, 0 \leqslant nk \leqslant T). \tag{2.10}$$

where

$$K_B(t, f; X, U) := \inf_{g \in U} (\| f - g \|_X + tB(g)) \qquad (t \geqslant 0; f \in X) \tag{2.11}$$

is the so-called modified K-functional.

Proof. Let $f \in X$, $g \in U$ be arbitrary. The estimate (2.9) and the uniform boundedness of the operators $E(t)$ for $0 \leqslant t \leqslant T$ imply that

$$\begin{aligned} \| E_k^n f - E(nk) f \|_X &\leqslant \| E_k^n f - E_k^n g \|_X + \| E_k^n g - E(nk) g \|_X \\ &\quad + \| E(nk) g - E(nk) f \|_X \\ &\leqslant C_T k^{-\beta} \| f - g \|_X + nk C_T k^{\alpha-\beta} B(g) + C_T \| f - g \|_X \\ &\leqslant C_T k^{-\beta} \{ \| f - g \|_X + nk^{\alpha-1} B(g) \} \end{aligned}$$

for $0 < k \leqslant \delta$, $0 \leqslant nk \leqslant T$. Since the left side of this inequality is independent of $g \in U$, taking the infimum over all $g \in U$ yields (2.10) for all $f \in X$.

Thus the rate of convergence for arbitrary $f \in X$ depends essentially, apart from the factor $k^{-\beta}$, upon the behavior of the K-functional $K_B(t, f)$. It is a continuous and monotone decreasing function of t for $t \to 0+$ with

$$\lim_{t \to 0+} K_B(t, f) = 0 \qquad (f \in X) \tag{2.12}$$

(since U is dense in X). Moreover, one has by definition (2.11)

$$\begin{aligned} K_B(t, f; X, U) &\leqslant \| f \|, & f \in X \\ &\leqslant tB(f), & f \in U. \end{aligned} \tag{2.13}$$

In many particular cases, such as $X = L^p(\mathbb{R})$, $1 \leqslant p < \infty$, with suitable U and $B(f)$, $K_B(t, f)$ can be estimated by the modulus of continuity (see below).

2.2. *The Main Theorem*

THEOREM 1. *Under the main hypotheses of Proposition 1 the following assertions are equivalent;*

(a) *the family $\{E_k\}$ is stable of order $\beta \geqslant 0$,*

(b) $$\| E_k^n f - E(nk) f \|_X \leqslant C_T k^{-\beta} K_B(nk^{\alpha+1}, f\colon X, U) \quad (0 < k \leqslant \delta, 0 \leqslant nk \leqslant T; f \in X),$$

(c,i,ii) $$\| E_k^n f - E(nk) f \|_X \leqslant C_T k^{-\beta} \begin{cases} M(f), & f \in X \\ nk^{\alpha+1} B(f), & f \in U \end{cases} \quad (0 < k \leqslant \delta; 0 \leqslant nk \leqslant T),$$

$M(f)$ being a constant depending on f.

Proof. By Prop. 1 one has (a) $\Rightarrow$ (b). The implication (b) $\Rightarrow$ (c) follows by (2.13). Concerning the step (c, i) $\Rightarrow$ (a), first note that by the uniform boundedness theorem (c, i) implies that there exists a constant C_T such that

$$\| E_k^n f - E(nk) f \|_X \leqslant C_T k^{-\beta} \| f \|_X \quad (0 < k \leqslant \delta, 0 \leqslant nk \leqslant T; f \in X).$$

This yields, the $E(t)$ being uniformly bounded for $0 \leqslant t \leqslant T$, for any $f \in X$,

$$\begin{aligned} \| E_k^n f \|_X &\leqslant \| E_k^n f - E(nk) f \|_X + \| E(nk) f \|_X \\ &\leqslant C_T k^{-\beta} \| f \|_X + C_T \| f \|_X \\ &\leqslant C_T k^{-\beta} \| f \|_X \qquad (0 < k \leqslant \delta, 0 \leqslant nk \leqslant T), \end{aligned}$$

so that there is stability of order β. This proves the theorem.

Comparing the above theorem with that of Lax, we see that the assertions of ordinary stability and pure convergence are replaced here by assertions (a) and (b), respectively, which involve orders. Our theorem therefore contains the Lax theorem in this sense since assertion (b) in case $\beta = 0$ implies the convergence assertion (2.7) in view of (2.12). Note that (2.10) or (b) of Theorem 1 gives a unified description of the (rate of) convergence covering both $f \in X$ and $f \in U$ (as is formulated separately in assertion (c)).

In concrete cases the consistency condition (2.5) is often given in a more suitable form, namely

$$\| [E_k - E(k)] E(t) f \|_X \leqslant C_T k^{1+\alpha} \| f \|_{D(A^{r+1})} \quad (0 < k \leqslant \delta; 0 \leqslant t \leqslant T; f \in D(A^{r+1})), \tag{2.14}$$

for some fixed $r \in \mathbb{N}$, usually with $\alpha = r$. Here A is the infinitesimal generator of the semigroup $\{E(t)\}_{t>0}$, and the domain $D(A^{r+1})$ of the $(r+1)$th power of A is a Banach space under the norm

$$\| f \|_{D(A^{r+1})} = \| f \|_X + \| A^{r+1} f \|_X \qquad (f \in D(A^{r+1})).$$

It is known (see [8, p. 12]) that $D(A^{r+1})$ is dense in X, and this is our subspace U of above. In this case the K-functional takes on the form

$$K(t, f; X, D(A^{r+1})) = \inf_{g \in D(A^{r+1})} \{\| f - g \|_X + t \| g \|_{D(A^{r+1})}\} \qquad (t \geqslant 0; f \in X). \tag{2.15}$$

Instead of (2.10) one then has the estimate

$$\| E_k^n f - E(nk) f \|_X \leqslant C_T k^{-\beta} K(k^r, f; X, D(A^{r+1}))$$
$$(0 < k \leqslant \delta, 0 \leqslant nk \leqslant T; f \in X). \tag{2.16}$$

Note that in the necessity part of the proof of Theorem 1 the consistency condition was not used (just as for the original Lax result). The question therefore arises whether convergence implies stability as well as consistency. In this direction we have

Proposition 2. *Given a properly posed problem* (2.1), (2.2) *such that* $f \in U := D(A^{r+1})$ *for some* $r \in \mathbb{N}$, *and a family of finite difference operators* $\{E_k\}_{k>0} \subset [X]$. *If the family is convergent in the sense*

$$\| E_k^n f - E(nk) f \|_X \leqslant C_T k^{-\beta} \cdot \begin{cases} M(f) & f \in X \\ nk^{\alpha+1} \| f \|_{D(A^{r+1})}, & f \in D(A^{r+1}) \end{cases}$$
$$(0 < k \leqslant \delta; 0 \leqslant nk \leqslant T), \quad (2.17,\text{i,ii})$$

then it is stable of order β *and consistent of order at least* $\alpha - \beta$ *on* $D(A^{r+1})$, *i.e.*,

$$\| [E_k - E(k)] E(t) f \|_X \leqslant C_T k^{\alpha-\beta+1} \{\sup_{0 \leqslant \omega \leqslant T} \| E(\omega) \|_{[X]}\} \| f \|_{D(A^{r+1})}$$
$$\textit{for} \quad 0 < k \leqslant \delta, \quad 0 \leqslant t \leqslant T, \quad \textit{and} \quad f \in D(A^{r+1}). \tag{2.18}$$

Proof. The fact that condition (2.17,i) implies that $\{E_k\}$ is stable of order β was already established in Theorem 1. Concerning consistency, the hypothesis (2.17, ii) for $n = 1$ with $f \in D(A^{r+1})$ replaced by $E(t) f \in D(A^{r+1})$ gives

$$\| E_k E(t) f - E(k) E(t) f \|_X \leqslant C_T k^{\alpha-\beta+1} \| E(t) f \|_{D(A^{r+1})}$$
$$\leqslant C_T k^{\alpha-\beta+1} \{\sup_{0 \leqslant \omega \leqslant T} \| E(\omega) \|_{[X]}\} \| f \|_{D(A^{r+1})}$$

for all $f \in D(A^{r+1})$, $k \in (0, \delta]$, $t \in [0, T]$. This is just the definition of consistency of order $\alpha - \beta$.

Whether the estimate (2.18) is the best possible one under our present definitions of convergence, consistency, and stability remains unsolved. If hypothesis (2.17, i, ii) is given with $\beta = 0$, then Proposition 2 yields consistency of order α. By the way, Spijker [24] (see also [2, p. 75]) has shown by an example that convergence of order $\alpha = 0$ does not necessarily imply consistency of order zero.

It is to be emphasized that convergence or consistency does not necessarily have the same meaning by all authors. Thus Chartres and Stepleman [10, 11] modified their definitions in such a fashion that in their form the Lax theorem even states that convergence implies stability as well as consistency. See also Spijker [25] in this matter.

3. Stability and Convergence in $L^p(\mathbb{R})$, $C(\mathbb{R})$ for a Certain Difference Scheme

Let X be one of the spaces $L^p(\mathbb{R})$, $1 \leqslant p < \infty$, or $C(\mathbb{R})$ (= set of uniformly continuous and bounded functions on $\mathbb{R}$) for $p = \infty$, with

$$\| v \|_X = \left(\int_{\mathbb{R}} | v(x)|^p \, dx \right)^{1/p}, \qquad X = L^p(\mathbb{R})$$
$$= \sup_{x \in \mathbb{R}} | v(x)|, \qquad X = C(\mathbb{R}).$$

Let us consider the initial-value problem

$$\begin{aligned} \frac{\partial}{\partial t} u &= \frac{\partial}{\partial x} u \qquad (x \in \mathbb{R}, t \geqslant 0) \\ u(x, 0) &= v(x) \qquad (x \in \mathbb{R}, v \in X). \end{aligned} \tag{3.1}$$

The genuine solution of (3.1) is given by $u(x, t) = [E(t) v](x) = v(x + t)$. The operators $E(t)$ build a semigroup of class (C_0) in $[X]$. For, the actual semigroup property is obvious; the (C_0)-property follows by continuity in the mean, i.e.,

$$\lim_{t \to 0+} \| E(t) v - v \|_X = \lim_{t \to 0+} \| v(\cdot + t) - v(\cdot)\|_X = 0 \qquad (v \in X).$$

Moreover, the set $U = \{v \in X: (d/dx) v \in X\}$ is dense in X, so that the problem (3.1) is properly posed in accordance with definition (2.3).

To obtain an approximate solution of (3.1) consider a finite difference operator E_k defined by

$$[E_k v](x) = \sum_{j=-\infty}^{\infty} a_j v(x - jh), \quad \sum_j | a_j | < \infty, \quad kh^{-1} = \lambda = \text{const} > 0. \tag{3.2}$$

To discuss the stability of E_k in X consider, following Brenner and Thomée [4], the characteristic function of E_k defined by

$$a(y) = \sum_{j=-\infty}^{\infty} a_j e^{-ijy} \qquad (y \in \mathbb{R}). \tag{3.3}$$

Now it is a known fact that for the space $L^2(\mathbb{R})$ a necessary and sufficient condition for stability (with $\beta = 0$) is that $|a(y)| \leqslant 1, y \in \mathbb{R}$. For $X = L^p(\mathbb{R})$ with $p \neq 2$ this condition is still necessary but not sufficient. If $|a(y)| < 1$ for some δ with $0 < |y| < \delta$, one can write for small y

$$a(y) = \exp(-i\lambda y + \psi(y)) \qquad (y \in \mathbb{R}),$$

where (unless $\psi \equiv 0$ and E_k is exact)

$$\psi(y) = \psi_0 y^{r+1}(1 + o(1)) \qquad (y \to 0),\ \psi_0 \neq 0,\ r \geqslant 1 \tag{3.4}$$

$$\operatorname{Re} \psi(y) = -\gamma y^s(1 + o(1)) \qquad (y \to 0),\ y > 0. \tag{3.5}$$

Here r and s can be interpreted as the orders of consistency and dissipation, respectively, of the operator E_k.

Brenner and Thomée [4] obtained the following estimate for stability in terms of these constants r and s: there are positive constants c_1 and c_2 such that for any $k > 0$, $n \in \mathbb{N}$,

$$c_1 n^{|(1/2)-(1/p)|(1-((r+1)/s))} \leqslant \| E_k^n \|_{[X]} \leqslant c_2 n^{|(1/2)-(1/p)|(1-((r-1)/s))}. \tag{3.6}$$

According to definition (2.8) this means that the family $\{E_k\}$ is stable of order $\beta = |(\frac{1}{2} - (1/p)| \cdot (1 - ((r + 1)/s))$ in $X = L^p(\mathbb{R})$ (or $X = C(\mathbb{R})$ for $p = \infty$).

To apply our theory we still have to verify the consistency condition of type (2.14). So we must estimate the expression $|[E_k v - E(k) v](x)|$ for $v \in X$ and small k. In many particular cases of (3.2) a Taylor expansion delivers an estimate of the following form

$$|[E_k v - E(k) v](x)| \leqslant Ck^r \left\{ \int_{x-k}^{x+k} | v^{(r+1)}(t)| \, dt \right\}. \tag{3.7}$$

The Hölder-inequality applied to the term in the curly brackets yields for $1 \leqslant p < \infty$:

$$|[E_k v - E(k) v](x)| \leqslant Ck^{r+1-(1/p)} \left\{ \int_{x-k}^{x+k} | v^{(r+1)}(t)|^p \, dt \right\}^{1/p}.$$

Integration over $-\infty < x < \infty$ and the Fubini theorem give

$$\int_{\mathbb{R}} |[E_k v - E(k)\, v](x)|^p \, dx \leqslant C k^{p(r+1)-1} \int_{\mathbb{R}} \int_{x-k}^{x+k} |\, v^{(r+1)}(t)|^p \, dt \cdot dx$$

$$\leqslant C k^{p(r+1)} \, \| \, v^{(r+1)}(\cdot)\|_{L^p}^p \, .$$

Thus we have the desired consistency estimate with $A = \partial/\partial x$ when replacing $v(x)$ by $[E(t)\, v](x)$, and noting that $\| \, E(t)\, v \, \|_{D(A^{r+1})} \leqslant C_T \, \| \, v \, \|_{D(A^{r+1})}$, $0 \leqslant t \leqslant T$, namely

$$\|[E_k - E(k)]\, E(t)\, v \, \|_{L^p} \leqslant C_T k^{r+1} \, \| \, v \, \|_{D(A^{r+1})} \, . \tag{3.8}$$

The case $X = C(\mathbb{R})$ is treated similarly.

We have now verified the hypotheses of Theorem 1 under the modifications of (2.14)–(2.17). In order to give our results a more concrete form, we make use of the known fact (see [8, p. 192]) that the K-functional given by (2.15) can be estimated by

$$\begin{aligned} C_{1,r}\,\omega_r(k, f; X) &\leqslant K(k^r, f; X, D(A^{r+1})) \\ &\leqslant C_{2,r}(k^r \, \| f \|_X + \omega_r(k, f; X)) \qquad (0 < k \leqslant \delta), \end{aligned} \tag{3.9}$$

where $C_{1,r}$, $C_{2,r}$ are constants independent of k and f, and

$$\omega_r(t, f; X) = \sup_{0<h<t} \left\| \sum_{s=0}^{r} \binom{r}{s} (-1)^{r-s} f(\cdot + sh) \right\|_X \quad (t \geqslant 0, r \in \mathbb{N}; f \in X) \tag{3.10}$$

is the rth modulus of continuity of $f \in X$.

This estimate may also be represented in terms of the socalled generalized Lipschitz spaces, giving the connection to the investigations by Thomée, Brenner, etc. $\mathrm{Lip}(\gamma, r, \infty; X)$ is the space of all elements $f \in X$ for which the norm

$$\| f \|_{\mathrm{Lip}(\gamma, r, \infty; X)} = \| f \|_X + \sup_{t>0} \{ t^{-\gamma} \omega_r(t, f; X) \} \tag{3.11}$$

is finite for some $\gamma \in \mathbb{R}$ and $r \in \mathbb{N}$ with $0 < \gamma < r$. This space is a Banach space under the norm (3.11), which is "intermediate" to X and $D(A^{r+1})$ in the sense that

$$D(A^{r+1}) \subset \mathrm{Lip}(\gamma, r, \infty; X) \subset X$$

with continuous embedding (the extreme cases X and $D(A^{r+1})$ being taken on for $\gamma = 0$ and $\gamma = r$, respectively). If $\gamma = l + \eta$, $0 < \eta \leqslant 1$, $l = 0, 1,\ldots, r-1$, an equivalent norm for this space is given by

$$\|f\|_X + \sup_{t>0}\{t^{-\eta}\omega_1(t, A^l f; X)\}, \qquad 0 < \eta < 1$$

$$\|f\|_X + \sup_{t>0}\{t^{-1}\omega_2(t, A^l f; X)\}, \qquad \eta = 1$$

(the rth modulus having been "reduced" to the more practical first or second modulus of continuity, respectively). In this terminology $K(t, f)$ given by (3.9) may be estimated by

$$\begin{aligned} K(k^r, f; X, D(A^{r+1})) &\leqslant Ck^\gamma[k^{r-\gamma}\|f\|_X + k^{-\gamma}\omega_r(k, f; X)] \\ &\leqslant Ck^\gamma \|f\|_{\mathrm{Lip}(\gamma, r, \infty; X)} \end{aligned}$$

where $0 < \gamma < r$, $0 < k \leqslant \delta$.

In view of (3.6) Theorem 1 yields

THEOREM 2. *Given the initial-value problem* (3.1) *in* $X\,(=L^p(\mathbb{R}), 1 \leqslant p < \infty$, *or* $C(\mathbb{R}))$ *and a finite difference approximation* E_k *defined by* (3.2) *with suitable constants* r *and* s *given by* (3.4), (3.5) *for which an expansion of type* (3.7) *is possible. The following assertions are valid:*

(a) *The family* $\{E_k\}_{k>0}$ *is stable of order*

$$\beta = |(\tfrac{1}{2}) - (1/p)|(1 - ((r+1)/s)) \text{ in } X,$$

(b)
$$\begin{aligned} \|E_k^n v - E(nk)v\|_X &\leqslant C_T k^{-\beta}[k^r \|v\|_X + \omega_r(k, v; X)] \\ &\qquad (0 < k \leqslant \delta, 0 \leqslant nk \leqslant T; v \in X) \\ &\leqslant C_T k^{-\beta} k^\gamma \|v\|_{\mathrm{Lip}(\gamma, r, \infty; X)} \\ &\qquad (v \in \mathrm{Lip}(\gamma, r, \infty; X)), \end{aligned}$$

(c)
$$\|E_k^n v - E(nk)v\|_X \leqslant C_T k^{-\beta} \cdot \begin{cases} M(v), & v \in X \\ k^r(\|v\|_X + \|v^{(r+1)}\|_X), & v^{(r+1)} \in X. \end{cases}$$

Assertion (b) gives us for initial data v possessing certain smoothness conditions estimates for the rate of convergence which lie between the extreme values $-\beta$ and $r - \beta$ for $0 < \beta < r$, convergence in the sense of (2.6) taking place if $\gamma > \beta$. These estimates are "intermediate" to the two cases of assertion (c).

For a particular difference operator take the Lax–Wendroff operator which is defined for (3.1) by

$$[E_k v](x) = (\tfrac{1}{2})(\lambda^2 + \lambda)\, v(x+h) + (1 - \lambda^2)\, v(x) + (\tfrac{1}{2})(\lambda^2 - \lambda)\, v(x-h)$$

with $\lambda = k/h = \text{const} > 0$. Elementary calculations give for its characteristic function the representations

$$a(y) = 1 - \lambda^2(1 - \cos y) + i\lambda \sin y,$$

$$|a(y)|^2 = 1 - \lambda^2 (1 - \lambda^2)(1 - \cos y)^2.$$

Therefore we have (ordinary) stability in $L^2(\mathbb{R})$ for $0 < \lambda < 1$. The constants r and s in (3.4), (3.5), (3.8) can here be shown to be $r = 2$ and $s = 4$. The Lax–Wendorff operator is therefore stable of order $\beta = \frac{1}{4}|\frac{1}{2} - (1/p)|$ in $L^p(\mathbb{R})$. In case $X = L^1(\mathbb{R})$ or $C(\mathbb{R})$ one has $\beta = \frac{1}{8}$, for $L^4(\mathbb{R})$ $\beta = 1/16$, and for $L^2(\mathbb{R})$ $\beta = 0$ as expected.

From assertion (b) we conclude for this instance that the finite difference approximation of (3.1) by (3.2) converges with order $\gamma - \beta$ provided the initial value v belongs to the space $\mathrm{Lip}(\gamma, 2, \infty: X)$ for some $0 < \gamma < 2$, i.e., if $\sup_{t>0} \{t^{-\gamma}\omega_2(t, v: X)\} < +\infty$.

It is to be noted that Brenner and Thomée obtained better estimates than the above in the case α is bounded from below: $(r + 1)|(\frac{1}{2}) - (1/p)| < \alpha < r + 1$. Then they have convergence of order $\alpha(1 - (1/(r + 1)))$, whereas ours is $\alpha(1 - (1/(r + 1))) - \beta$, β being the order of stability. This is to be expected since they used intricate and long estimates depending upon methods of Fourier analysis for this particular example: we on the other hand just applied our general theorem. Finally note that the approximation estimate of Prop. 1 can be improved for holomorphic semigroup operators; see Weis [33].

ACKNOWLEDGMENTS

The authors would like to thank Karl Scherer (Bonn) for his interest in the early stages of this work, particular in connection with Weis [33]; Henning Esser (Aachen) for a critical reading of the manuscript and the suggestion to add Proposition 2, as well as Rolf J. Nessel (Aachen) for most valuable advice. The contribution by R. Weis was supported by DFG Research Grant Bu 166/23.

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